Radiative zone solar magnetic fields and g-modes

T. I. Rashba[1,2]⋆, V. B. Semikoz[2,3]† and J. W. F. Valle[3]‡

[1] *Max-Planck-Institut für Physik (Werner-Heisenberg-Institut), Föhringer Ring 6, D-80805 München, Germany*
[2] *Pushkov Institute of Terrestrial Magnetism, Ionosphere and Radiowave Propagation of the Russian Academy of Sciences,*
IZMIRAN, Troitsk, Moscow region, 142190, Russia
[3] *AHEP Group, Instituto de Física Corpuscular – C.S.I.C./Universitat de València,*
Edificio Institutos de Paterna, Apt 22085, E-46071, València, Spain

5 February 2008

ABSTRACT

We consider a generalized model of seismic-wave propagation that takes into account the effect of a central magnetic field in the Sun. We determine the g-mode spectrum in the perturbative magnetic field limit using a one-dimensional Magneto-Hydrodynamics (MHD) picture. We show that central magnetic fields of about 600-800 kG can displace the pure g-mode frequencies by about 1%, as hinted by the helioseismic interpretation of GOLF observations.

Key words: MHD – Sun: helioseismology – Sun: interior – Sun: magnetic fields.

1 INTRODUCTION

Currently there is very little direct information about the structure and strength of magnetic fields in the radiative zone (RZ) of the Sun, for a short review see Introduction of the paper (Burgess et al. 2004a). Some authors argue that for the young Sun ($\lesssim 3 - 30$ Myr) relatively small fields, $\lesssim 10$ kG (Moss 2003) and $\lesssim 1$ Gauss (Kitchatinov, Jardine & Collier-Cameron 2001) could survive, being relic fields captured from the primordial ones in the protostar plasma. For the Sun at the present epoch there is an upper bound of $2 - 3$ MG near the tachocline obtained from the magnetic splitting of acoustic oscillations (Ruzmaikin & Lindsey 2002). However, some authors have considered very strong magnetic fields in the RZ, up to 30 MG (Couvidat, Turck-Chièze & Kosovichev 2003).

Here we suggest a new way to estimate the magnetic field strengths in the RZ of the Sun by relating them to the frequency shifts of g-mode candidates suggested by the first observations made with the GOLF (Global Oscillations at Low Frequencies) experiment (Turck-Chièze et al. 2004). We discuss some effects of RZ magnetic fields which could explain the displacement of g-mode frequencies with respect to the theoretical frequencies calculated in the absence of magnetic field. Indeed the existence of such shifts are hinted in GOLF's data. If eventually confirmed by further data, the idea that RZ magnetic fields cause such frequency shifts would provide us with a useful tool to estimate their magnitude.

In order to find spectra of seismic waves accounting for the magnetic field in the RZ a number of assumptions is required. For example:

(i) We consider ideal MHD neglecting both the heat conductivity and viscosity contributions to energy losses, as well as the ohmic dissipation.

(ii) We linearize the MHD equations about a static background configuration, *i.e.* a background configuration which is time independent and for which the background fluid velocity vanishes, $\boldsymbol{v}_0 = 0$.

(iii) We assume the fluctuations to be adiabatic, with the contributions of fluctuations to the heat source vanishing: $Q' = 0$.

(iv) Moreover, we consider a fully ionized ideal gas, so that the thermodynamic quantity, first adiabatic exponent $\gamma = c_p/c_V$, is time independent and uniform. For numerical estimates we will take $\gamma = 5/3$ for hydrogen plasma.

(v) We adopt the Cowling approximation, which amounts to the neglect of perturbations of the gravitational potential, (i.e.: $\phi' = 0$).

(vi) We assume a rectangular geometry with Cartesian coordinates: x, y, and z, where z corresponds to the solar radial direction. The background quantities vary along the z direction only (which implies the local gravitational acceleration, $\mathbf{g}$, is directed along the z axis, but in opposite direction). We also take a constant, uniform background magnetic field, $\mathbf{B}_0$, pointing along the x axis.

(vii) The background mass-density profile is assumed to be exponential, $\rho_0 = \rho_c \exp[-z/H]$, for constant ρ_c and H. The conditions of hydrostatic equilibrium for the background then determine the profiles of thermodynamic quantities, and in particular imply γ is a constant. We assume

that the Brunt-Väisälä frequency is zero in the convective zone (CZ) and non-zero, but constant in the RZ.

In what follows, we shall again specify the assumptions used, as they are needed, in order to keep clear which results rely on which assumptions.

Note that, deep within the radiative zone, the last approximation above holds to very good approximation for real mass-density profiles obtained by Standard Solar Models, provided we identify the z direction with the radial direction. The constancy of γ in this region is also expected since the highly-ionized plasma satisfies an ideal gas equation-of-state to good approximation. The rectangular geometry provides a reasonable approximation so long as we do not examine too close to the solar centre. What is important about our choice for $\mathbf{B}_0$ is that it is slowly varying in the region of interest, and it is perpendicular to both $\mathbf{g}$ and all background gradients, $\nabla\rho_0$, ∇p_0, *etc.*

As suggested in (Burgess et al. 2004a) such one-Dimensional (1-D) picture can be fully described in analytical terms in contrast to the 3-D case. There are two parameters which describe the spectra of magneto-gravity waves (Burgess et al. 2004a): (i) strength of the background magnetic field B_0 and (ii) the dimensionless transversal wave number $K = k_x H$. Here H is the density scale height and k_x is the projection of the wavevector onto the x axis. Let us estimate the value of the transversal wave number that could be relevant for the g-mode candidates observed on the photosphere.

Since g-modes decay in the CZ as $\sim e^{-Kz/H}$, only modes with low transversal wave number $K \sim 1 - 4$ (long wave lengths) could be seen at the photosphere. This follows from the simple estimate for the longitudinal fluid velocity $v_z(z)$ which is directed along the Sun-Earth line and causes the Doppler shifts of optic lines registered by the GOLF experiment:

$$v_z(z = R_\odot) = \frac{NH}{K}\frac{b_z}{B_0}e^{-3K} \simeq 2 \text{ mm sec}^{-1}. \qquad (1)$$

This formula comes from Eq. (14) (equivalent to our Eq. (10)) and Eq. (30) of Ref. (Burgess et al. 2004a) for the decaying solution $B_z^{(1)}(z) = b_z e^{-Kz/H}$, where $B_z^{(1)}$ is the z-component of the magnetic field perturbation.

Here in the right hand side we substituted the sensitivity of the GOLF-instrument to the minimum fluid speed, $v_z = 2$ mm/sec, while in the left hand side we substituted the frequency estimate $\omega \sim N$ and the wave length through the CZ: $R_\odot - z_{RZ} = 3H = 0.3R_\odot$. For instance, substituting for the magnetic field perturbation, $b_z/B_0 = 0.01$, $N = 2.8 \times 10^{-3}$ rad sec^{-1} for the Brunt-Väisälä frequency in the RZ, $H = 0.1R_\odot = 7 \times 10^9$ cm (Bahcall 1988) for the density scale height, one obtains $e^{-3K}/K \simeq 10^{-6}$, from which the estimate $K = K_{max} \sim 4$ comes.

We organize our presentation as follows. In Section 2 we formulate the MHD model for an ideal plasma. In Section 3 we linearize the full set of MHD equations and then derive a single master equation for the z-component of the perturbation velocity $v_z^{(1)}(z)$. This component leads to the Doppler shift of the optic frequencies measured in helioseismic experiments. In Subsection 3.1 we check the validity of the master equation against the well-known case of standard helioseismology in an isotropic plasma, without mag-

netic fields. In Subsection 3.2 we derive the simplified master equation in the perturbative limit. In this limit there are no MHD (slow or Alfvén) resonances within the Sun, a situation which was treated in (Burgess et al. 2004a). This perturbative method allows us to use a standard quantum mechanical 1-D approach to determine an exact analytical spectrum of g-modes in the presence of RZ magnetic fields. In Section 4 we summarize our results.

2   BASIC IDEAL MHD EQUATIONS

We describe the Sun as an ideal hydrodynamical system characterized by nonlinear MHD equations. The mass conservation law for the total density $\bar{\rho}$ can be written as

$$\frac{\partial\bar{\rho}}{\partial t} + \nabla\cdot(\bar{\rho}\mathbf{v}) = 0. \qquad (2)$$

The total pressure $\bar{\rho}$ is a sum of equilibrium, ρ_0, and non-equilibrium, ρ, parts, $\bar{\rho} = \rho_0 + \rho$. Since viscosity can be neglected, momentum is conserved according to

$$\frac{\partial\mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} = -\frac{1}{\bar{\rho}}\nabla\bar{P} + \frac{1}{4\pi\bar{\rho}}(\mathbf{B}\cdot\nabla)\mathbf{B} + \mathbf{g}. \qquad (3)$$

Here the total pressure $\bar{P} = P_0(z) + P$ consists of two terms: the equilibrium and non-equilibrium parts. The first is expressed as $P_0(z) = p_0(z) + B_0^2/8\pi$ and obeys the equation $\nabla P_0 = \rho_0(z)\mathbf{g}$. The non-equilibrium part of the total pressure $P = p + (1/4\pi)(\mathbf{B}_0\cdot\mathbf{B}') + \mathbf{B}'^2/8\pi$ involves non-linear terms coming from the total magnetic field $\mathbf{B} = \mathbf{B}_0 + \mathbf{B}'$.

The evolution of the magnetic field is governed by Faraday's equation,

$$\frac{\partial\mathbf{B}}{\partial t} = (\mathbf{B}\cdot\nabla)\mathbf{v} - (\mathbf{v}\cdot\nabla)\mathbf{B} - \mathbf{B}u, \qquad (4)$$

where $u = \nabla\cdot\mathbf{v} \neq 0$ is the compressibility of the gas. Finally the conservation of entropy leads to the energy conservation law for an ideal plasma,

$$\frac{\partial}{\partial t}\left(\frac{\bar{p}}{\bar{\rho}^\gamma}\right) + \mathbf{v}\cdot\nabla\left(\frac{\bar{p}}{\bar{\rho}^\gamma}\right) = 0, \qquad (5)$$

where $\bar{p} = p_0 + p$ is the total gas pressure.

3   LINEAR MHD MASTER EQUATION

For definiteness here we consider the same generalized helioseismic model already proposed in Ref. (Burgess et al. 2004a), adopting an approximately rectangular rather than cylindrical, geometry. In this case it is convenient to choose a Cartesian coordinate system whose z-axis is the "radial" direction; opposite to the local acceleration, $\mathbf{g} = (0, 0, -g(z))$. With this choice we take z in the range $(0, R_\odot)$, where $z = 0$ represents the solar center and $z = R_\odot$ denotes the solar surface. The radiative zone corresponds to $z \lesssim 0.7R_\odot$.

The model assumes the background magnetic field to be directed along the x-axis $\mathbf{B_0} = (B_0(z), 0, 0)$, ensuring that physical gradients lie along the z-axis. Such a field mimics both a toroidal field lying in the equatorial plane and a poloidal field perpendicular to the equatorial plane.

We now linearize the above MHD equations (2-5), so that all variables are split into background and fluctuating

Euler quantities, $f = f_0 + f^{(1)}$, with $f^{(1)}$ denoting small fluctuations about the background value f_0. In what follows we also neglect rotation of the Sun, so that the background velocity is zero, $v_0 = 0$. Moreover, we assume that the MHD perturbations are independent of y, which implies that $B_y^{(1)} = v_y^{(1)} = 0$, leaving just six perturbation functions, instead of eight.

In addition, following (Ruderman, Hollweg & Goossens 1997), we consider plane waves propagating along the x-axis and therefore the dependence of all functions on the coordinates (x, z, t) can be reduced to just $\theta \equiv x - Vt$, with $V = \omega/k_x$ being the phase velocity. This can be done since all functions depend only on the harmonic factor $e^{ik_x x - i\omega t}$.

This way all perturbations are functions of two variables, $f^{(1)}(\theta, z)$, and the linearized MHD equations involve partial derivatives over θ and z. Notice that $\partial/\partial t = -V\partial/\partial\theta$ and $\partial/\partial x = \partial/\partial\theta$.

Let us rewrite the initial system for the six functions $f^{(1)}(\theta, z)$ in Eqs. (2-5) expressing all of them through just two quantities, $v_z^{(1)}(z, \theta)$ and $P^{(1)}(z, \theta)$. One obtains:

$$\frac{\partial P^{(1)}}{\partial z} = \frac{\rho_0 D_A}{V}\frac{\partial v_z^{(1)}}{\partial \theta} - \rho^{(1)}g, \tag{6}$$

$$\frac{\partial v_z^{(1)}}{\partial z} = \frac{V}{F}\frac{\partial P^{(1)}}{\partial \theta} + \frac{gV^2}{D_C}v_z^{(1)}, \tag{7}$$

$$\frac{\partial \rho^{(1)}}{\partial \theta} = \frac{V^2}{D_C}\frac{\partial P^{(1)}}{\partial \theta} + \frac{v_z^{(1)}}{V}\frac{d\rho_0}{dz} + \frac{\rho_0 g D_A}{D_C V}v_z^{(1)}. \tag{8}$$

$$\frac{\partial B_x^{(1)}}{\partial \theta} = \frac{B_0(V^2 - c_s^2)}{\rho_0 D_C}\frac{\partial P^{(1)}}{\partial \theta} + \frac{v_z^{(1)}}{V}\frac{dB_0}{dz} + \frac{gB_0 V}{D_C}v_z^{(1)}, \tag{9}$$

$$B_z^{(1)} = -B_0\frac{v_z^{(1)}}{V}, \tag{10}$$

$$\frac{\partial v_x^{(1)}}{\partial \theta} = \frac{Vc_s^2}{\rho_0 D_C}\frac{\partial P^{(1)}}{\partial \theta} - \frac{gv_A^2}{D_C}v_z^{(1)}. \tag{11}$$

We introduced above the important coefficients F, D_A, D_C which in turn are functions of "z" via $\rho_0(z)$, the Alfvén velocity $v_A(z)$, and the sound velocity $c_s(z)$:

$$F = \frac{\rho_0 D_C}{V^2 - c_s^2}, \quad D_A = V^2 - v_A^2,$$

$$D_C = (v_A^2 + c_s^2)(V^2 - c_T^2), \tag{12}$$

where $c_T^2 = v_A^2 c_s^2/(v_A^2 + c_s^2)$ is the squared cusp velocity. The zeros of the last coefficient D_C correspond to the slow ($V = c_T$) or Alfvén ($V \approx v_A$, $v_A \ll c_s$) resonances respectively.

Differentiating Eq. (6) over θ and using the next two equations one gets the master equation for the function $v_z^{(1)}(\theta, z)$ as:

$$\frac{\partial}{\partial z}F\frac{\partial}{\partial z}v_z^{(1)} - \rho_0 D_A\frac{\partial^2 v_z^{(1)}}{\partial \theta^2} -$$
$$- \left[g^2\frac{F}{D_C} + \rho_0\frac{dg}{dz} + g\frac{d}{dz}\left(\frac{Fc_s^2}{D_C}\right)\right]v_z^{(1)} = 0. \tag{13}$$

It is easy to see that, neglecting gravity, our generalized equations (6)-(12) recover the total system (34)-(38) derived in (Ruderman, Hollweg & Goossens 1997) and Eq. (13) coincides with Eq. (39) in (Ruderman, Hollweg & Goossens 1997). In case of the constant gravity, g=const, while the sound velocity $c_s^2(z) = \gamma gH(z)$ depends on the

varying height scale $H(z)$, Eq. (13) recovers Eq. (7) in (Miles, Allen & Roberts 1992).

For simplicity we consider below the particular case of uniform and constant magnetic field, B_0=const, constant gravity, g=const, and density scale height, H=const. Therefore the sound speed c_s and the Brunt-Väisälä frequency N are also constants in RZ, $c_s^2 = \gamma gH$=const, $N^2 = (g/H)(\gamma - 1)/\gamma$=const. This is somewhat of a simplification of the real Sun, but it allows us to derive qualitative estimates of the magnetic field corrections to the pure g-mode spectrum.

In what follows in our numerical estimates we will use the following values: $\gamma = 5/3$ (hydrogen plasma), $N = 2.8 \times 10^{-3}$ rad sec^{-1}, $H = 0.1R_\odot = 7 \times 10^9$ cm (Bahcall 1988), hence $g = 1.37 \times 10^5$ cm s^{-2}, $c_s = 4 \times 10^7$ cm/s.

Separating in master equation (13) the exponential dependence $e^{ik_x\theta}$ and taking into account the background density profile $\rho_0 = \rho_c e^{-z/H}$, one gets the ordinary second order equation[1]:

$$\left[1 - \frac{v_A^2}{V^2}\left(1 - \frac{V^2}{c_s^2}\right)\right]\frac{d^2 v_z^{(1)}}{dz^2} - \frac{1}{H}\frac{dv_z^{(1)}}{dz} +$$
$$+ k_x^2\left(\frac{N^2}{\omega^2} - \left(1 - \frac{V^2}{c_s^2}\right)\left[1 - \frac{v_A^2}{V^2}\right]\right)v_z^{(1)} = 0. \tag{14}$$

This equation generalizes Eq. (16) of our work (Burgess et al. 2004a) accounting for the compressibility parameter $a_1 = V^2/c_s^2 = \omega^2/k_x^2 c_s^2 (= 0.24\omega^2/K^2 N^2$ for hydrogen plasma with $\gamma = 5/3$). Note that a_1 is small for very low frequencies of g-modes $\omega \ll N$, $a_1 \ll 1$, and therefore it was neglected in the problem considered in (Burgess et al. 2004a).

3.1 Zero magnetic field limit

For $v_A = B_0 = 0$ with the use of the transformation $v_z^{(1)}(z) = \Psi(z)e^{z/2H}$ Eq. (14) converts into

$$\frac{d^2\Psi}{dz^2} + \left[\frac{\omega^2}{c_s^2} - \frac{1}{4H^2} + k_x^2\left(\frac{N^2}{\omega^2} - 1\right)\right]\Psi = 0. \tag{15}$$

This 1-D MHD equation coincides with the 3-D oscillation Eq. (7.90) in (Christensen-Dalsgaard 2003):

$$\frac{d^2\Psi}{dr^2} + \left[\frac{\omega^2}{c_s^2} - \frac{1}{4H^2} + k_h^2\left(\frac{N^2(r)}{\omega^2} - 1\right)\right]\Psi = 0, \tag{16}$$

here we have substituted the acoustical cut-off frequency $\omega_c = (c_s^2/4H^2)(1 - 2dH/dr) \approx c_s^2/4H^2$ and put $S_l^2 = l(l + 1)c_s^2/r^2 \simeq k_h^2 c_s^2$, see Eq. (4.60) in (Christensen-Dalsgaard 2003).

We would like to stress that, in the JWKB approximation for low frequency g-modes, $c_s^2 \gg V^2$, $S_l^2 \gg \omega^2$, both equations (15) and (16) lead to the analogous spectra:

$$\omega_g(n, k_x) = \frac{k_x N}{\pi(n + 1/2)}z_{RZ} \quad \text{(1D model)},$$

$$\omega_g(n, l) = \frac{\sqrt{l(l+1)}\int_{r_1}^{r_2} N(r)dr/r}{\pi(n + l/2 + \alpha_g)} \quad \text{(3D model)}, \tag{17}$$
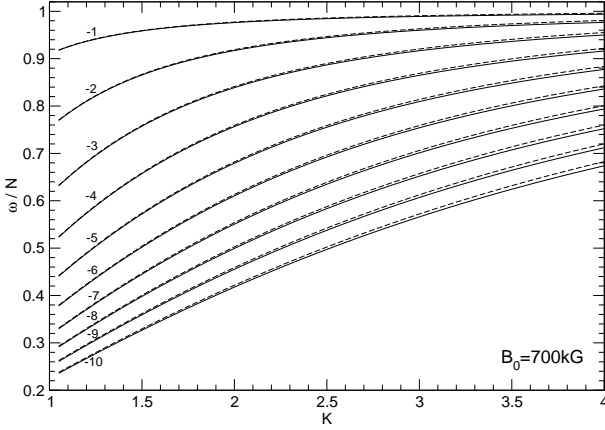
Figure 1. The g-mode frequencies for $n = -1, -2, \ldots, -10$ normalized to Brunt-Väisälä frequency, ω/N, versus wave number, K. Solid lines are for zero magnetic field, while dashed lines correspond to a fixed magnetic field $B_0 = 700$kG.

Figure 2. The resonance positions are shown in the plane $B_0 - K$ (magnetic field *vs* wave number along magnetic field). The curves separate the regions where perturbative or non-perturbative approaches may or should be used. $\zeta_z = 1$ means that at place z the Alfvén resonance occurs.

where n is the radial order of the wave, and the transversal wave number k_x is the 1-D analogue of the angular degree l in the 3-D case.

Let us denote the factor in brackets in Eq. (15) as $\beta^2/4H^2$,

$$\beta = \sqrt{4K^2\left(\frac{N^2}{\omega^2} - 1\right) - 1 + 4K^2 a_1}, \qquad (18)$$

here $K = k_x H$ is the dimensionless wave number, $K \geqslant K_{min} \simeq 2\pi H/R_\odot \sim 1$. Eq. (15) can be rewritten using Eq. (18) as

$$\frac{\mathrm{d}^2\Psi}{\mathrm{d}z^2} + \frac{\beta^2}{4H^2}\Psi(z) = 0. \qquad (19)$$

In RZ where $N_{RZ} \equiv N = const \neq 0$ one obtains the solution of Eq. (19) in the form $\Psi_{RZ}(z) = C_{RZ}\sin(\beta z/2H)$, where C_{RZ} is a constant. It accounts for the boundary condition $v_z^{(1)}(0) = 0$ at the center of the Sun. In CZ we approximate Brunt-Väisälä frequency by the value $N_{CZ} = 0$. In that case β tends to $i\Gamma = i\sqrt{4K^2(1 - a_1) + 1}$. Taking into account a second boundary condition, that there are no solutions which grows with z, one gets the decaying MHD wave in the form: $\Psi_{CZ}(z) = C_{CZ}\exp(-z\Gamma/2H)$.

Now matching both solutions at the top of RZ, $z = z_{RZ}$, (for logarithmic derivatives see the book by Landau & Lifshitz (2000)), one obtains the dispersion equation in the case $B_0 = 0$:

$$\beta \cot\left[\frac{\beta z_{RZ}}{2H}\right] = -\Gamma, \qquad (20)$$

or the g-mode spectrum in our 1-D model is given by $\beta z_{RZ}/2H + \pi n = \mathrm{arccot}[-\Gamma/\beta]$, $n = -1, -2, -3, \ldots$ (see solid curves in the Fig. 1).

3.2 Magnetic corrections to the g-mode spectrum

In order to obtain spectra of g-modes in the presence of magnetic field let us define the coefficient in front of the second order derivative in Eq. (14) as $1 - \zeta$, where $\zeta = v_{A0}^2 e^{z/H}(1 - a_1)/V^2$. The Alfvén velocity at the solar center
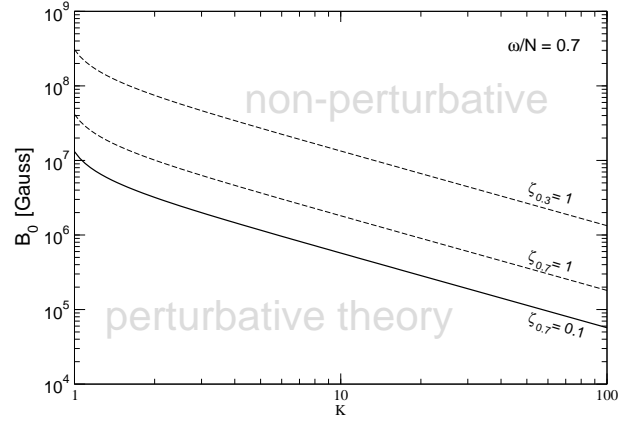
is given by $v_{A0} = B_0/\sqrt{4\pi\rho_c} \approx (B_0/43.4 \text{ Gauss}) \text{ cm sec}^{-1}$, here, and through the paper, $\rho_c = 150 \text{ g/cm}^3$.

We consider the perturbative regime for magnetic fields, where $v_A^2 = v_{A0}^2 e^{z/H} \ll V^2 = \omega^2/k_x^2 = [N^2 H^2/K^2](\omega/N)^2$, so that $\zeta \ll 1$ and MHD resonances ($\zeta = 1$) do not appear within the RZ. This region is the lower one in Fig. 2. The dashed curves labelled $\zeta_{0.3} = 1$ and $\zeta_{0.7} = 1$ correspond to resonances that occur at $0.3R_\odot$ and $0.7R_\odot$ in the non-perturbative region. The solid curve is chosen to illustrate the separation between the two regimes, according to the criterion $\zeta_{0.7} = \zeta(0.7R_\odot) = 0.1 \ll 1$. The perturbative magnetic field for which the maximum Alfvén velocity is small, $v_A^2(0.7R_\odot) \ll V^2$, obeys

$$B_0 \ll \frac{2.8 \times 10^7}{K} \text{ Gauss (for } \omega \lesssim \text{N).} \qquad (21)$$

The absence of MHD resonances allows us to set

$$v_z^{(1)} \equiv \left[1 - \frac{v_A^2}{V^2}\left(1 - \frac{V^2}{c_s^2}\right)\right]^{-1/2} e^{z/2H}\Psi(z),$$

and thus derive from Eq. (14) the $O(v_A^2/V^2)$-correction to Eq. (19):

$$\frac{\mathrm{d}^2\Psi}{\mathrm{d}z^2} + \left[\frac{\beta^2}{4H^2} + k_B^2\right]\Psi(z) = 0, \qquad (22)$$

here the wave number correction k_B^2 is given by

$$k_B^2 = (1 - a_1)\frac{v_A^2}{V^2}k_x^2\left[\frac{N^2}{\omega^2} + a_1\right].$$

Introducing the notation:

$$A_0 = (1 - a_1)\frac{v_{A0}^2}{V^2}\left[\frac{N^2}{\omega^2} + a_1\right] > 0, \qquad (23)$$

one gets from Eq. (22):

$$\frac{\mathrm{d}^2\Psi}{\mathrm{d}z^2} + \left[\frac{\beta^2}{4H^2} + k_x^2 A_0 e^{z/H}\right]\Psi(z) = 0$$

and after the change $2s = z/H + \ln(4K^2 A_0)$ this equation takes the form

$$\frac{\mathrm{d}^2\Psi}{\mathrm{d}s^2} + [\beta^2 + e^{2s}]\Psi(s) = 0. \tag{24}$$

The general solution of Eq. (24) is expressed via the Bessel functions of the first kind, $\Psi(s) = C_1 J_{i\beta}(e^s) + C_2 J_{-i\beta}(e^s)$ (see Gradstein & Ryzhik (2000)), C_1 and C_2 are constants. Now, coming back to the variable z, we use the boundary condition at the solar center, $\Psi(0) = 0$ $(v_z(0) = 0)$, to obtain the solution of Eq. (24) in RZ, $0 \leqslant z \leqslant z_{RZ}$,

$$\Psi_{RZ}(z) = C_{RZ}\left[J_{i\beta}(2KA_0^{1/2}e^{z/2H}) - \right.$$
$$\left. - \frac{J_{i\beta}(2KA_0^{1/2})}{J_{-i\beta}(2KA_0^{1/2})} J_{-i\beta}(2KA_0^{1/2}e^{z/2H})\right]. \tag{25}$$

In CZ the Brunt-Väisälä frequency N vanishes, $N = 0$, so that we are led to the same solution as in isotropic case $\Psi_{CZ}(z) = C_{CZ}\exp(-z\Gamma/H)$, neglecting the possible existence of a CZ magnetic field (if such a field is present, with strength of 300 kG, the g-mode frequencies change by no more than 10^{-5}). Since the arguments of Bessel functions in Eq. (24) are small, we can use the first terms in the Bessel function series only,

$$J_\nu(z) = \left(\frac{z}{2}\right)^\nu \sum_{k=0}^\infty \frac{(-1)^k}{k!\Gamma(\nu+k+1)}\left(\frac{z}{2}\right)^{2k}, \quad |arg\ z| < \pi,$$

$$J_{\pm i\beta}(2KA_0^{1/2}e^{z/2H}) \approx \frac{(KA_0^{1/2}e^{z/2H})^{\pm i\beta}}{\Gamma(1\pm i\beta)} \times$$
$$\times \left[1 - \frac{K^2 A_0 e^{z/H}}{1\pm i\beta} + O(K^2 A_0 e^{z/H})\right], \tag{26}$$

Then by matching the logarithmic derivatives of the solutions Ψ_{RZ} and Ψ_{CZ} at the top of RZ, $z = z_{RZ}$, one obtains the generalized dispersion equation for the case $B_0 \neq 0$:

$$\beta\left(1 + \frac{2\kappa_{RZ}^2}{1+\beta^2}\right)\left(1 - \beta\frac{2\kappa_{RZ}^2}{(1+\beta^2)}\frac{1}{\sin(\beta z_{RZ}/H)}\right) \times$$
$$\times \cot\left[\beta\frac{z_{RZ}}{2H}\right] - \frac{2\kappa_{RZ}^2}{1+\beta^2} = -\Gamma, \tag{27}$$

where $\kappa_{RZ} = KA_0^{1/2}e^{z_{RZ}/2H}$.

This is our main equation. Clearly, when the magnetic field correction tends to zero, hence $2\kappa_{RZ}^2/(1+\beta^2)$ tends to 0, one recovers the isotropic case, given in Eq. (20).

We search for a perturbative solution of Eq. (27) of the form $\beta = \beta_0(1+\delta_B)$ and $\omega = \omega_0(n)(1+\alpha_B(n))$ where $\beta = \beta_0$ and correspondingly $\omega = \omega_0(n)$ are the solution of Eq. (20) for $B_0 = 0$ and the smallness of $\delta_B \ll 1$ and $\alpha_B(n) \ll 1$ follows from $\kappa_{RZ}^2 \ll 1$.

The g-mode spectra for fixed $B_0 = 700$ kG are shown on Fig. 1 (see dashed curves). Figs. 3 and 4 display our results for α_B as a function of mode radial order n, magnetic field and wave number K. One sees that the magnetic field shift α_B of a g-mode frequency $\omega_0(n)$ is always positive, $\alpha_B > 0$. In Fig. 3 we show the absolute values of the shift $\alpha_B(n)$ for different g-modes and for fixed $B_0 = 700$ kG as a function of K. Conversely, in Fig. 4 we fix the wave number $K = 2$ and plot $\alpha_B(n)$ as a function of B_0. One sees that, the higher the mode radial order $|n|$, the less the magnetic field strength required to produce a given g-mode frequency shift.
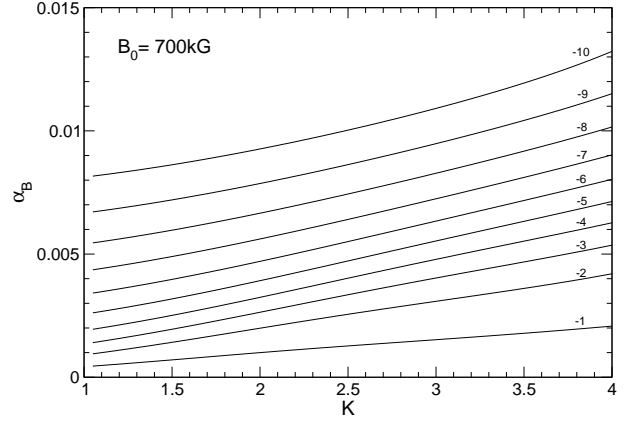

Figure 3. The shift of g-mode frequency α_B versus wave number, K, shown for fixed magnetic field strength, $B_0 = 700$ kG and different mode radial orders, $n = -1, -2, \ldots, -10$.
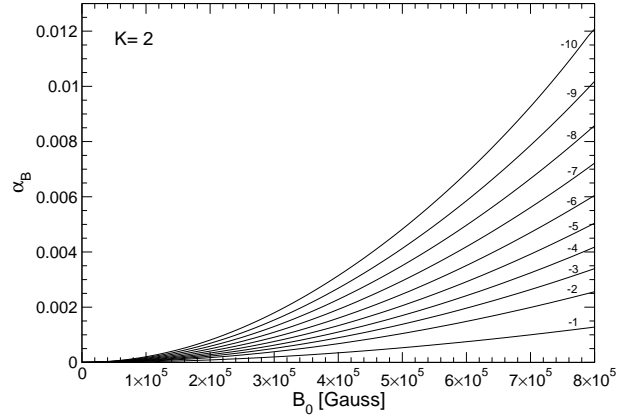

Figure 4. The shift of g-mode frequency α_B versus magnetic field strength, B_0, shown for fixed wave number, $K = 2$, and different mode radial orders, $n = -1, -2, \ldots, -10$.

4 DISCUSSION

We have given a generalization of helioseismology to account for the presence of central magnetic fields in the Sun. We have determined the resulting g-mode spectrum within the framework of a perturbative one–dimensional Magneto-Hydrodynamics model.

There are three factors influencing g-mode observation in helioseismic experiments. First, such g-modes should have long wave lengths to penetrate the CZ: in our case $K \leqslant 4$, and in the 3D-model, low values of l. Second, the radial order n should also be low, otherwise, such low frequency g-modes are much below the present experimental sensitivity. Third, there is a strong influence of the RZ magnetic field.

If the magnetic field is too strong (more than a few MG) all g-modes are locked within the Alfvén cavity (see Burgess et al. (2004a)), hence these g-modes decay far beneath the CZ becoming invisible in the photosphere. This happens because the MHD energy in the radial direction is fully diverted to the transversal plane at the Alfvén resonant layer position. On the other hand, for very small magnetic fields ($B_0 \ll 1$ MG) the magnetic field effect is negligible and can not account for a possible discrepancy between present experimental data and theoretical predictions

of seismic models. In contrast, we have shown that a solar radiative zone magnetic fields of intermediate magnitude, of the order 600-800 kG, can displace the pure g-mode frequencies by about 1% with respect to our model of seismic wave propagation, a value close to what is hinted by results of the GOLF experiment.

We find that higher modes require a smaller magnetic field to produce a given g-mode frequency shift. However encouraging this result may sound, let us stress again that in our simple one-dimensional MHD picture (with further assumptions, such as adiabatic oscillations, exponential density profile, constant gravity, etc) we can only make a qualitative estimate of the magnetic field corrections to the pure g-mode spectrum. For example, our formalism can not explain the magnetic splitting of g-mode frequencies over azimuthal number m, as it requires 3-D. Further work in 3-D geometry is necessary to perform a quantitative comparison with the frequency patterns observed in the GOLF experiment (Turck-Chièze et al. 2004). Even within the simple analytic approach 1-D MHD model one may include viscosity effects in non-ideal plasma with finite conductivity, and also take into account magnetic field diffusion stabilizing MHD instabilities.

Last, but not least, recall that our perturbative analysis avoids the appearance of MHD resonances that could lead to density spikes. These are potentially important, as they can affect neutrino propagation through the solar RZ (Burgess et al. 2003, 2004a,b). Improved determination of neutrino mixing parameters, e.g. by Kam-LAND (Eguchi et al. 2003), allows one to carry out neutrino tomography in deep solar interior. Both regimes of "magnetized helioseismology" (i) the MHD seismic models and (ii) the analysis of MSW neutrino oscillations in noisy Sun are complementary tools to explore RZ magnetic fields. A fully quantitative analysis may require the inclusion of the differential rotation in the RZ (Turck-Chièze et al. 2005) as well as non-linearities.

Note added: As this paper was being typed, we saw a paper by Hasan, Zahn, & Christensen-Dalsgaard (2005) (astro-ph/0511472), where a similar idea to probe the internal magnetic field of slowly pulsating B-stars through g modes is given. Their 3-D results are consistent with our simpler 1-D estimates.

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